УДК 519.21

Козаченко Ю.¹, д.ф.-м.н., проф.
Островський Є.², проф.
Сирота Л.³, проф.

**Еквівалентність між розподілами
хвостів, нормою в лебегових просторах
та орлічевою нормою для випадкових
величин, для яких не виконується
умова Крамера**

¹Кафедра теорії ймовірностей, статистики
та актуарної математики, Київський націо-
нальний університет імені Тараса Шевченка,
м. Київ, Україна,

e - mails: yvk@univ.kiev.ua, ykoz@ukr.net

² Кафедра математики, Бар-Іланський універ-
ситет, 59200, Рамат Ган, Ізраїль,
e-mail: eugostrovsky@list.ru

³ Кафедра математики, Бар-Іланський універ-
ситет, 59200, Рамат Ган, Ізраїль,
e - mail: sirota3@bezeqint.net

Kozachenko Yu.¹, Dr.Sci., Prof.,
Ostrovsky E.², Prof.,
Sirota L.³, Prof.

**Equivalence between tails, Grand
Lebesgue Spaces and Orlicz norms for
random variables without Kramer's
condition**

¹Department of Probability Theory, Statisti-
cs and Actuarial Mathematics, Taras Shevchenko
National University of Kyiv, Kiev, Ukraine,

e - mails: yvk@univ.kiev.ua, ykoz@ukr.net

² Department of Mathematics, Bar-Ilan Universi-
ty, 59200, Ramat Gan, Israel.

e-mail: eugostrovsky@list.ru

³ Department of Mathematics, Bar-Ilan Universi-
ty, 59200, Ramat Gan, Israel.

e - mail: sirota3@bezeqint.net

*В роботі розглядаються неасимптотичні попарні з точністю до мультиплікативної кон-
станти взаємозв'язки між поведінкою хвостів, моментами (норми в просторі Лебега) та нор-
мою Орліча для випадкових величин, для яких, в загальному випадку, не виконується умова
Крамера.*

*Ключові слова: Випадкова величина та випадковий вектор (r.v.), метод сідлової точки,
хвіст розподілу, моменти, простори Лебега-Ріца, Орліча і простори Лебега (GLS); повільно
змінюючі функції, перетворення Юнга-Фенчеля, теорема і нерівність Фенхеля-Моро, функція
Юнга-Орліча, оцінка Маркова - Чернова, неасимптотичні оцінки, умова Крамера.*

*Ключові слова: імпульсна перехідна функція, лінійна однорідна система, гауссовий процес,
сумісна корелограма.*

*We offer in this paper the non-asymptotical pairwise bilateral exact up to multiplicative constants
interrelations between the tail behavior, moments (Grand Lebesgue Spaces) norm and Orlicz's norm for
random variables (r.v.), which does not satisfy in general case the Kramer's condition.*

*Key Words: Random variable and random vector (r.v.), saddle - point method, tail of distributi-
on, moments, Lebesgue - Riesz, Orlicz and Grand Lebesgue Spaces (GLS); slowly varying functions,
Young-Fenchel transform, theorem and inequality of Fenchel-Moreau, Young-Orlicz function, Markov -
Chernov's estimate, non-asymptotical estimates, Kramer's condition.*

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

Communicated by Assoc. Prof. Rozora I.V.

1 Definitions. Notations. Previous results. Statement of problem.

Let (Ω, B, P) be certain probability space with non - trivial probability measure P and correspondent expectation E , and let $\xi = \xi(\omega)$, $\omega \in \Omega$ be numerical valued random variable. We denote as usually by $|\xi|_p$, $p \in [1, \infty]$

its classical Lebesgue - Riesz $L_p = L(p) = L_p(\Omega)$ norm

$$|\xi|_p := [E|\xi|^p]^{1/p}, \quad 1 \leq p < \infty;$$

$$|\xi|_\infty := \text{vraisup}_{\omega \in \Omega} |\xi(\omega)|.$$

The so - called *tail - function* $T_\xi(u)$, $u \geq 0$ for

this random variable ξ is defined by a formula

$$T_{\xi}(u) \stackrel{\text{def}}{=} \max \{ \mathbf{P}(\xi > u), \mathbf{P}(\xi < -u) \}, \quad u \geq 0. \quad (1.1)$$

An equivalent version:

$$\overline{T}_{\xi}(u) := \mathbf{P}(|\xi| > u), \quad u \geq 0. \quad (1.1a)$$

Obviously,

$$\overline{T}_{\xi}(u) \leq T_{\xi}(u) \leq 2 \overline{T}_{\xi}(u), \quad u \geq 0,$$

the equivalence.

The aim of this report is to establish the reciprocal non - asymptotic interrelations separately mutually possibly exact up to multiplicative constant between tail functions, suitable Orlicz and Grand Lebesgue Spaces norms for random variables.

We do not suppose that the considered in this article r.v. satisfy the famous Kramer's condition:

$$\exists \epsilon_0 > 0 \quad \forall \lambda : |\lambda| < \epsilon_0 \Rightarrow \mathbf{E} \exp(\lambda \xi) < \infty,$$

in contradiction with previous works, see, for example, works [2], [6], [7], [8], [19].

Throughout this paper, the letters $C, C_j(\cdot)$ etc. will denote a various positive finite constants which may differ from one formula to the next even within a single string of estimates and which does not depend on the essentially variables p, x, λ, y, u etc.

We make no attempt to obtain the best values for these constants.

The immediate predecessor of offered report is the article [7], in which was considered the case when the considered r.v. satisfy the famous Kramer's condition. See also [6], [8], chapters 1,2 etc.

We will use in this report in general at the same techniques as in [7].

Recall that the so - called Young-Fenchel transform $g \rightarrow g^*$ of arbitrary real valued function $g = g(x)$ is defined as follows

$$g^*(y) \stackrel{\text{def}}{=} \sup_{x \in \text{Dom}(g)} (xy - g(x)). \quad (1.2)$$

The symbol $\text{Dom}(g)$ denotes as ordinary the domain of definition (in particular, finiteness) of the function $g(\cdot)$.

Let us bring some used further examples. Define the function

$$\begin{aligned} \phi_{m,L} &= \phi_{m,L}(\lambda) := m^{-1} \lambda^m L(\lambda), \\ \lambda &> 0, \quad m = \text{const} > 1, \end{aligned} \quad (1.3)$$

where $L = L(\lambda)$ is positive *slowly varying* at infinity, i.e. as $\lambda \rightarrow \infty$ function. Then as $x \rightarrow \infty$

$$\phi_{m,L}^*(x) \sim (m')^{-1} x^{m'} L^{-1/(m-1)} \left(x^{1/(m-1)} \right), \quad (1.4)$$

and as ordinary for arbitrary value $m > 1$

$$m' \stackrel{\text{def}}{=} \frac{m}{m-1}.$$

If for instance $L(\lambda) = [\ln(\lambda + e)]^r$, $r = \text{const} \in \mathbf{R}$, i.e.

$$\phi(\lambda) := \phi_{m,r}(\lambda) = m^{-1} \lambda^m [\ln(\lambda + e)]^r,$$

$$\lambda \geq 0, m = \text{const} > 1, r \in \mathbf{R},$$

then as $x \rightarrow \infty$

$$\phi_{m,r}^*(x) \sim (m')^{-1} x^{m'} [\ln(x + e)]^{-r/(m-1)}, \quad (1.5)$$

see, e.g. [18], p. 40 - 42.

Analogously if

$$\phi(\lambda) := \phi_{m,r,q}(\lambda) = m^{-1} \lambda^m [\ln(\lambda + e)]^r [\ln \ln(\lambda + e^e)]^q,$$

$$\lambda > 0, m = \text{const} > 1, r, q \in \mathbf{R},$$

then similarly as $x \rightarrow \infty$

$$\begin{aligned} \phi_{m,r,q}^*(x) &\sim (m')^{-1} x^{m'} [\ln(x + e)]^{-r/(m-1)} \times \\ &\times [\ln \ln(x + e^e)]^{-q/(m-1)}. \end{aligned} \quad (1.6)$$

More generally, if

$$L(\lambda) = [\ln \lambda]^r M(\ln \lambda),$$

where $M = M(\lambda)$ is positive slowly varying function as $\lambda \rightarrow \infty$, then as $x \rightarrow \infty$

$$\begin{aligned} \phi_{m,L}^*(x) &\sim (m')^{-1} x^{m'} [\ln x]^{-r/(m-1)} \times \\ &\times M^{-1/(m-1)}(\ln x). \end{aligned} \quad (1.7)$$

The case $m = 1$ is more complicated. Define the Ψ function $\psi^{(L)} = \psi^{(L)}(p)$ as follows

$$\psi^{(L)}(p) \stackrel{\text{def}}{=} \frac{p}{L(p)}, \quad (1.8)$$

where as before $L = L(\lambda)$ is positive continuous slowly varying function as $\lambda \rightarrow \infty$ tending to infinity as $\lambda \rightarrow \infty$. Let also the r.v. ξ be from the Grand Lebesgue Space $G\psi^{(L)}$ with unit norm:

$$\|\xi\|_{G\psi^{(L)}} \stackrel{\text{def}}{=} \sup_{p \geq 1} \left\{ \frac{|\xi|_p}{\psi^{(L)}(p)} \right\} = 1,$$

then by the direct definition of these norms

$$|\xi|_p \leq \frac{p}{L(p)}, \quad p \geq 1. \quad (1.9)$$

We deduce by means of Chebyshev - Markov inequality

$$T_\xi(x) \leq \exp(-C(L) x \ln L(x)). \quad (1.10)$$

Conversely, let the estimate (1.10) be a given for some r.v. ξ and some such a positive function $L = L(\cdot)$ for which $L(x) \uparrow \infty$, then

$$|\xi|_p \leq C(L) \frac{p}{L(p)}, \quad p \geq 1. \quad (1.11)$$

2 Grand Lebesgue Spaces (GLS).

Let $(\Omega, B, \mathbf{P})$ be again at the same probability space. Let also $\psi = \psi(p)$, $p \in [1, b)$, $b = \text{const} \in (1, \infty]$ (or $p \in [1, b]$) be certain bounded from below: $\inf \psi(p) > 0$ continuous inside the *semi-open* interval $p \in [1, b)$ numerical function such that the function

$$h(p) = h[\psi](p) \stackrel{\text{def}}{=} p \ln \psi(p) \quad (2.0)$$

is convex.

An important example. Let η be a random variable such that there exists $b = \text{const} > 1$ so that $|\xi|_b < \infty$. The *natural* $G\Psi$ function $\psi_\eta = \psi_\eta(p)$ for the r.v. η is defined by a formula

$$\psi_\eta(p) \stackrel{\text{def}}{=} |\eta|_p.$$

We can and will suppose $b = \sup\{p, \psi(p) < \infty\}$, so that $\text{supp } \psi = [1, b)$ or $\text{supp } \psi = [1, b]$. The set of all such a functions will be denoted by $\Psi(b) = \{\psi(\cdot)\}$; $\Psi := \Psi(\infty)$.

We will consider in this article only the case when $b = \infty$; i.e. $\Psi := \Psi(\infty)$.

By definition, the (Banach) Grand Lebesgue Space (GLS) space $G\psi = G\psi(b)$ consists on all the numerical valued random variables (measurable functions) ζ defined on our measurable space and having a finite norm

$$\|\zeta\| = \|\zeta\|_{G\psi} \stackrel{\text{def}}{=} \sup_{p \in [1, b)} \left\{ \frac{|\zeta|_p}{\psi(p)} \right\}. \quad (2.1)$$

The function $\psi = \psi(p)$ is named *generating function* for the Grand Lebesgue Spaces.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1, 2; and were investigated in particular in many works, see e.g. [3], [4], [5], [6], [7], [8], [15], [16], [17]. We refer here some used in the sequel facts about these spaces and supplement more.

It is known that if $\zeta \neq 0$, and $\zeta \in G\psi(b)$, then

$$T_\zeta(y) \leq \exp(-h_\psi^*(\ln(y/\|\zeta\|))), \quad y \geq \|\zeta\|, \quad (2.2)$$

where

$$h(p) = h[\psi](p) \stackrel{\text{def}}{=} p \ln \psi(p), \quad 1 \leq p < b.$$

Namely, let $\|\zeta\|_{G\psi(b)} = 1$; therefore by means of Chebyshev - Markov inequality

$$T_\zeta(y) \leq \frac{\psi^p(p)}{y^p} = \exp(-p \ln y + p \psi(p)),$$

following

$$T_\zeta(y) \leq \inf_{p \in [1, b)} \exp(-p \ln y + p \psi(p)) =$$

$$= \exp(-h[\psi]^*(\ln(y/\|\zeta\|))), \quad y \geq e \cdot \|\zeta\|.$$

Conversely, the last inequality may be reversed in the following version: if the r.v. ζ satisfies the *Kramer's condition* and

$$\mathbf{P}(|\zeta| > y) \leq \exp(-h_\psi^*(\ln(y/K))). \quad y \geq e \cdot K,$$

$$K = \text{const} \in (0, \infty),$$

and if the function $h_\psi(p)$, $1 \leq p < \infty$ is positive, continuous, convex and such that

$$\lim_{p \rightarrow \infty} \psi(p)/p = 0,$$

then $\zeta \in G\psi$, herewith $\|\zeta\| \leq C(\psi) \cdot K$ and conversely

$$\|\zeta\|G\psi \leq C(\psi)K \leq C_2(\psi)\|\zeta\|G\psi,$$

$$0 < C_1(\psi) < C_2(\psi) < \infty. \quad (2.3)$$

Introduce the following *exponential* Young - Orlicz function

$$N_\psi(u) = \exp(h_\psi^*(\ln|u|)), \quad |u| \geq 1;$$

$$N_\psi(u) = Cu^2, \quad |u| < 1,$$

and the correspondent Orlicz norm will be denoted by $\|\cdot\|L(N_\psi) = \|\cdot\|L(N)$. It was done

$$\begin{aligned} \|\zeta\|G\psi &\leq C_1\|\zeta\|L(N) \leq \\ &\leq C_2\|\zeta\|G\psi, \quad 0 < C_1 < C_2 < \infty. \end{aligned} \quad (2.4)$$

If for instance

$$\psi(p) = \psi_m(p) \stackrel{\text{def}}{=} p^{1/m}, \quad p \in [1, \infty),$$

where $m = \text{const} > 1$, then

$$0 \neq \xi \in G\psi_m \Leftrightarrow T_\xi(u) \leq \exp(-C(m)u^m).$$

Define also the correspondent Young - Orlicz function

$$N_m(u) := \exp(|u|^m), \quad |u| \geq 1;$$

$$N_m(u) = Cu^2, \quad |u| \leq 1.$$

The relation (2.3) means in addition in this case

$$\|\zeta\|G\psi_m \leq C_1(m)\|\zeta\|L(N_m) \leq C_2\|\zeta\|G\psi_m, \quad (2.5)$$

$$0 < C_1(m) < C_2(m) < \infty.$$

Notice that in the case when $m \in (0, 1)$ the correspondent random variable ξ does not satisfy the Kramer's condition. We intend to generalize the last propositions further on the case just in particular $m \in (0, 1]$.

Define as an example the following *degenerate* $G\psi$ function

$$\psi_{(r)}(p) = 1, \quad 1 \leq p \leq r;$$

$$\psi_{(r)}(p) = \infty, \quad p > r; \quad r = \text{const} > 1.$$

The $G\psi_{(r)}$ norm of an arbitrary r.v. η is quite equivalent to the classical Lebesgue - Riesz L_r norm

$$\|\eta\|G\psi_{(r)} = |\eta|_r. \quad (2.6)$$

Thus, the Grand Lebesgue Spaces are direct generalizations of the Lebesgue - Riesz spaces.

3 Auxiliary estimates from the saddle - point method.

We must investigate in advance one interest and needed further integrals. Namely, let (X, M, μ) , $X \subset R$ be non-trivial measurable space with non-trivial sigma finite measure μ .

We assume at once $\mu(X) = \infty$, as long as the opposite case is trivial for us. We intend to estimate for λ greatest values of real parameter λ , say $\lambda > e$, the following integral

$$I(\lambda) := \int_X e^{\lambda x - \zeta(x)} \mu(dx). \quad (3.1)$$

assuming of course its convergence for all the sufficiently great values of the parameter λ . The offered below estimates may be considered as a some generalizations of the saddle - point method.

Here $\zeta = \zeta(x)$ is non-negative measurable function, not necessary to be convex.

We represent now two methods for *upper* estimate $I(\lambda)$ for sufficiently greatest values of the real parameter λ .

Note first of all that if in contradiction the measure μ is finite: $\mu(X) = M \in (0, \infty)$; then the integral $I(\lambda)$ allows a very simple estimate

$$\begin{aligned} I(\lambda) &\leq M \cdot \sup_{x \in X} \exp(\lambda x - \zeta(x)) = \\ &= M \cdot \exp(\zeta^*(\lambda)). \end{aligned} \quad (3.2)$$

Let now $\mu(X) = \infty$ and let $\epsilon = \text{const} \in (0, 1)$; let us introduce the following auxiliary integral

$$K(\epsilon) := \int_X e^{-\epsilon \zeta(x)} \mu(dx). \quad (3.3)$$

It will be presumed its finiteness at last for some positive value $\epsilon_0 \in (0, 1)$; then $\forall \epsilon \geq \epsilon_0 \Rightarrow K(\epsilon) < \infty$.

Then the following measures are probabilistic:

$$\nu_\epsilon(A) := \frac{\int_A \exp(-\epsilon \zeta(x)) \mu(dx)}{K(\epsilon)}, \quad \epsilon \geq \epsilon_0. \quad (3.4)$$

We have

$$\frac{I(\lambda)}{K(\epsilon)} = \int_X \exp(\lambda x - (1 - \epsilon)\zeta(x)) \nu_\epsilon(dx) \leq$$

$$\begin{aligned} & \exp\left\{\sup_{x \in X} [\lambda x - (1 - \epsilon)\zeta(x)]\right\} = \\ & = \exp\left\{(1 - \epsilon)\zeta^*\left(\frac{\lambda}{1 - \epsilon}\right)\right\}. \end{aligned}$$

Following,

$$I(\lambda) \leq K(\epsilon) \cdot \exp\left\{(1 - \epsilon)\zeta^*\left(\frac{\lambda}{1 - \epsilon}\right)\right\} \quad (3.5)$$

and hence:

Theorem 3.1 We assert actually under formulated here conditions, in particular, the condition of the finiteness of $K(\epsilon)$ for some value $\epsilon_0 \in (0, 1)$:

$$I(\lambda) \leq \inf_{\epsilon \in (0, 1)} \left[K(\epsilon) \cdot \exp\left\{(1 - \epsilon)\zeta^*\left(\frac{\lambda}{1 - \epsilon}\right)\right\} \right]. \quad (3.6)$$

We can detalized the choice of the value ϵ in the estimates (3.5) - (3.6). Namely, denote

$$\theta = \theta(\lambda) := \frac{c_1}{\lambda \zeta^{*'}(2\lambda)}, \quad \lambda \geq \lambda_0 = \text{const} > 0. \quad (3.7)$$

The value λ_0 is selected such that $\theta(\lambda) \leq 1/2$, $\lambda \geq \lambda_0$. Then

$$\frac{\lambda}{1 - \epsilon} \leq \lambda(1 + 2\epsilon),$$

and we have taking into account the convexity of the function $\zeta^*(\cdot)$ and denothing $\phi(\lambda) = \zeta^*(\lambda)$:

$$\phi\left(\frac{\lambda}{1 - \theta}\right) \leq \phi(\lambda + 2\lambda\theta) \leq$$

$$\phi(\lambda) + 2\theta\lambda \phi'(2\lambda) \leq c_2 + \phi(\lambda).$$

To summarize:

$$I(\lambda) \leq c_2 K(\theta(\lambda)) \exp(\zeta^*(\lambda)). \quad (3.8)$$

As regards the function $K = K(\theta(\lambda))$, note that if $X = R^+$, $\mu(dx) = dx$, and if

$$\zeta(x) \geq c_4 x, \quad x \geq 0, \quad (3.9)$$

then

$$K(\theta(\lambda)) \leq c_5 \lambda \zeta^{*'}(2\lambda),$$

hence

$$I(\lambda) \leq c_6 \lambda \zeta^{*'}(2\lambda) \cdot \exp(\zeta^*(\lambda)), \quad \lambda > \lambda_0. \quad (3.10)$$

If in turn instead (3.9) there holds

$$\zeta(x) \geq c_7 x^\alpha, \quad \alpha = \text{const} > 0,$$

$$X = R_+, \quad \mu(dx) = dx,$$

then

$$I(\lambda) \leq c_8 \left[\lambda \zeta^{*'}(2\lambda) \right]^{1/\alpha} \cdot \exp(\zeta^*(\lambda)), \quad \lambda > \lambda_0. \quad (3.11)$$

Theorem 3.2. Suppose in addition $X = (a, \infty)$, $a = \text{const} \in R$, or $X = R$, and that

$$\exists C = \text{const} \in (0, \infty), \exists \alpha = \text{const} > 1 \Rightarrow$$

$$\Rightarrow \zeta(x) \geq Cx^\alpha, \quad x \geq 1. \quad (3.12)$$

Then there exists a finite positive constant $C = C(\zeta, a)$ such that for sufficiently values λ , say for $\lambda \geq 1$

$$I(\lambda) \leq \exp(\zeta^*(C\lambda)). \quad (3.13)$$

Proof, in particular, the finiteness of $K(\epsilon)$, $\epsilon \in (0, 1)$ contains in fact in [9], chapter 2.1.

We represent now an opposite method, which was introduced in particular case in [7], [8], sections 1.2. Inded, let $\gamma = \text{const} \in (0, 1)$. We apply the Young's inequality

$$\lambda x \leq \zeta(\gamma x) + \zeta^*(\lambda/\gamma),$$

therefore

$$I(\lambda) \leq e^{\zeta^*(\lambda/\gamma)} \cdot \int_X e^{\zeta(\gamma x) - \zeta(x)} \mu(dx) =$$

$$= R(\gamma) e^{\zeta^*(\lambda/\gamma)}, \quad (3.14)$$

where

$$R(\gamma) := \int_X e^{\zeta(\gamma x) - \zeta(x)} \mu(dx). \quad (3.15)$$

We obtained really the following second estimate.

Lemma 3.2.

$$I(\lambda) \leq \inf_{\gamma \in (0,1)} \left[R(\gamma) e^{\zeta^*(\lambda/\gamma)} \right]. \quad (3.16)$$

4 Main results: connection between tail behavior and Grand Lebesgue Space norm.

Statement of problem: given a tail function $T_\xi(y)$ for the certain (non - zero) random variable ξ of the form

$$T_\xi(y) \leq \exp(-h^*[\psi](\ln y)), \quad y \geq 1, \quad (4.1)$$

where $\psi(\cdot) \in G\Psi$. It is required to prove $\xi \in G\psi$, or on the other words to obtain an estimate of the form $\|\xi\|G\psi < \infty$.

Recall that the inverse conclusion: $\|\xi\|G\psi = 1 \Rightarrow (4.1)$ is known, see (2.2).

So, let the estimate (4.1) be a given. We have for the values $p \geq e$

$$p^{-1} |\xi|_p^p \leq \int_0^\infty x^{p-1} \exp(-h^*[\psi](\ln x)) dx = \int_{-\infty}^\infty \exp(p y - h^*(y)) dy. \quad (4.2)$$

It remains to use the proposition of theorem 3.1.

Theorem 4.1. Suppose

$$C(h) := \sup_{p \in [1, \infty)} \left[h^{*'}[\psi](p) \right]^{1/p} < \infty. \quad (4.3)$$

If the r.v. ξ satisfies the inequalities (4.1) and (4.3), then $\xi \in G\psi$:

$$\|\xi\|G\psi \leq 2 C[h] e^{1/e} < \infty. \quad (4.4)$$

Proof. It is sufficient to note that the function $p \rightarrow h[\psi(p)]$ is continuous and convex and that

$$(h^*)^* = h^{**} = h$$

by virtue of theorem of Fenchel - Moreau.

Let us bring some examples.

Example 4.1. Put as before

$$\psi_m(p) = p^{1/m},$$

but here $m = \text{const} \in (0, \infty)$. Let $\xi \in G\psi_m$ and $\|\xi\|G\psi_m = 1$.

Note that in the case $m \in (0, 1)$ the r.v. ξ does not satisfy in general case the Kramer's condition. But we conclude on the basis of theorem 3.1 $\|\xi\|G\psi_m \in (0, \infty) \iff$

$$\exists C(m) \in (0, \infty), T_\xi(u) \leq \exp(-C(m) u^m), \quad u \geq 0. \quad (4.5)$$

More precisely, if $\|\xi\|G\psi_m = 1$, then

$$T_\xi(u) \leq \exp(-(me)^{-1} y^m), \quad y > 0.$$

Inversely, assume

$$T_\xi(u) \leq \exp(-y^m), \quad y > 0.$$

Then it follows from theorem 3.1

$$\|\xi\|G\psi_m \leq e^{m+1/e}$$

or equally

$$|\xi|_p \leq e^{m+1/e} p^{1/m}, \quad p \geq 1.$$

Let us consider a more general case, indeed, introduce as above the following Ψ function

$$\psi_{m,L}(p) \stackrel{\text{def}}{=} p^{1/m} L(p), \quad m = \text{const} > 0, \quad (4.6)$$

where $L = L(p)$, $p \geq 1$ is some positive continuous slowly varying as $p \rightarrow \infty$ function. We impose for simplicity the following condition on this function:

$$\forall \theta > 0 \Rightarrow \sup_{p \geq 1} \left[\frac{L(p^\theta)}{L(p)} \right] =: C(\theta) < \infty. \quad (4.7)$$

This condition is satisfied, if for example $L(p) = [\ln(p+1)]^r$, $r = \text{const}$.

It follows again from theorem 3.1 that the r.v. ξ belongs to the space $G\psi_{m,L}$:

$$\|\xi\|G\psi_{m,L} = \sup_{p \geq 1} \left[\frac{|\xi|_p}{\psi_{m,L}(p)} \right] = 1 \quad (4.8)$$

if and only if

$$T_{\xi}(y) \leq \exp(-C(m, L) y^m / L(y)), \quad y \geq e. \quad (4.9)$$

As a particular case: define the Ψ – function

$$\psi_{(m,r)}(p) := p^{1/m} \ln^r(p+1), \quad (4.10)$$

$$p \geq 1; \quad m = \text{const} > 0, \quad r = \text{const} \in \mathbb{R}.$$

The random variable ξ belongs to the space $G\psi_{(m,r)}$:

$$\begin{aligned} & \|\xi\| G\psi_{(m,r)} = \\ & = \sup_{p \geq 1} \left[\frac{|\xi|_p}{\psi_{(m,r)}(p)} \right] = l \in (0, \infty) \end{aligned} \quad (4.11a)$$

if and only if

$$\begin{aligned} T_{\xi}(u) & \leq \exp(-C(m, r) (u/l)^m \ln^{-r}(u/l)), \\ & u \geq e l. \end{aligned} \quad (4.11b)$$

Example 4.2. A boundary case.

We introduce the following $G\Psi$ function

$$\begin{aligned} \psi^{(s)}(p) & = p (\ln(p+1))^s, \\ s & = \text{const} \in \mathbb{R}, \quad p \in [1, \infty). \end{aligned} \quad (4.12a)$$

Then the non - zero r.v. ν belongs to the $G\psi^{(s)}$ space if and only if for $y \geq 0$

$$T_{\nu}(y) \leq \exp(-C(s) y \ln^{-s}(y+1)). \quad (4.12b)$$

Note that the r.v. ν satisfies the Kramer's condition if and only if $s \leq 0$. The case $s = 0$ correspondent to the exponential distribution for the r.v. ν ; the case $s = -1$ take place in particular when the r.v. ν has a Poisson distribution, which obey's but the exponential moments.

Example 4.3.

Let us consider the following $\psi_{\beta}(p)$ function

$$\psi_{\beta,C}(p) := \exp(Cp^{\beta}), \quad (4.13)$$

$C, \beta = \text{const} > 0$. Obviously, the r.v. τ for which

$$\forall p \geq 1 \Rightarrow |\tau|_p \geq \psi_{\beta,C}(p)$$

does not satisfy the Kramer's condition.

Let ξ be a r.v. belongs to the $G\psi_{\beta,C}(\cdot)$ space:

$$\|\xi\| G\psi_{\beta,C} = 1, \quad (4.14a)$$

or equally

$$|\xi|_p \leq \exp\{Cp^{\beta}\}, \quad p \in [1, \infty). \quad (4.14b)$$

The last restriction is quite equivalent to the following tail estimate

$$T_{\xi}(y) \leq \exp\left(-C_1(C, \beta) [\ln(1+y)]^{1+1/\beta}\right), \quad (4.15)$$

as $y > 0$.

5 Main results: connection between tail behavior and Orlicz's space norm.

We retain the notations and definitions of the previous sections, in particular,

$$G(u) = G[\psi](u) = h^*[\psi](\ln u), \quad \psi \in G\Psi \quad (5.0)$$

etc. Define also the following Young - Orlicz function $N[\psi](u) :=$

$$\exp[G(u)] = \exp[h^*[\psi](\ln |u|)], \quad u \geq e;$$

$$N[\psi](u) = C u^2, \quad |u| < e. \quad (5.1)$$

We will prove in this section that the tail estimate (2.2) of the r.v. ξ is completely equivalent under some simple conditions to the finiteness of its Orlicz's norm $\|\xi\| LN[\psi]$.

Recall that we do not suppose that the r.v. ξ satisfies the Kramer's condition.

Proposition 5.1. If for some r.v. ξ there holds $\|\xi\| LN[\psi] = K \in (0, \infty)$, then for $y \geq e \cdot \|\xi\|$

$$T_{\xi}(y) \leq \exp(-h_{\psi}^*(\ln(y/(C K)))) \quad (5.2)$$

Proof basing only on the Chebyshev - Markov inequality is at the same as before in the inequality (2.2), see [2], chapters 2,3; [6], [7], [19]. Namely, we deduce that for some positive finite constant C_1

$$\mathbf{E} \exp(G(|\xi|/C_1)) < \infty.$$

It remains to use the Chebyshev - Markov inequality.

Proposition 5.2. Assume in addition to the foregoing conditions on the function $\psi(\cdot)$ that the function $G[\psi](u)$ satisfies the following restriction:

$$\begin{aligned} \exists \alpha = \text{const} \in (0, 1), \exists K = \text{const} > 1, \\ \forall x \in (0, \infty) \Rightarrow G(x/K) \leq \alpha G(x). \end{aligned} \quad (5.3)$$

If for some r.v. ξ

$$T_\xi(y) \leq \exp(-h_\psi^*(\ln(y))), \quad y \geq e, \quad (5.4)$$

then the r.v. ξ belongs to the Orlicz space $LN[\psi]$:

$$\|\xi\|_{LN[\psi]} \leq C(\psi, \alpha, K) < \infty. \quad (5.5)$$

Proof is more complicated than one for proposition 5.1. It used the following auxiliary fact.

Lemma 5.1. Let a function $g: R_+ \rightarrow R_+$ be monotonically increasing, $T = T_\xi(x)$, $S = S_\eta(x)$, $x \geq 0$ be two tail functions correspondingly for non-negative r.v. ξ , η and such that

$$T_\xi(x) \leq S_\eta(x), \quad x \geq 0.$$

We assert:

$$\int_0^\infty g(x) |dT_\xi(x)| \leq \int_0^\infty g(x) |dS_\eta(x)|.$$

Proof of lemma 5.1. One can suppose without loss of generality that both the tail functions T and S are continuous and strictly decreasing. Further, one can realize both the r.v. ξ , η on the classical probability space $\Omega = \{\omega\} = [0, 1]$ equipped with ordinary Lebesgue measure:

$\xi = \xi(\omega) = (1-T)^{-1}(\omega)$, $\eta = \eta(\omega) = (1-S)^{-1}(\omega)$, where f^{-1} denotes the inverse function.

We have $\xi(\omega) \leq \eta(\omega)$ a.e., therefore $g(\xi) \leq g(\eta)$ a.e., and all the more so

$$\begin{aligned} \mathbf{E}g(\xi) &= - \int_0^\infty g(x) dT_\xi(x) \leq \\ &\leq - \int_0^\infty g(x) dS_\eta(x) = \mathbf{E}g(\eta), \end{aligned}$$

Q.E.D.

Proof of proposition 5.2. Let the pair of numbers (α, K) be from the condition (5.3). We have relating the proposition of Lemma 5.1 $\mathbf{E} \exp(G(\xi/K)) =$

$$\int_0^\infty \exp G(x/K) |dT_\xi(x)| \leq$$

$$\leq \int_0^\infty \exp G(x/K) |d \exp(-G(x))| \leq \quad (5.6)$$

$$\begin{aligned} &\int_0^\infty \exp[\alpha G(x)] |d \exp(-G(x))| = \\ &= \int_0^1 z^{-\alpha} dz = \frac{1}{1-\alpha} < \infty. \end{aligned} \quad (5.7)$$

We used by passing (5.6) $\rightarrow$ (5.7) the fact quite thin from an article [14]. It follows immediately from this estimates that $\xi \in L(N[\psi])$, see for example [19], p. 31 - 33.

Examples. The condition (5.3) is satisfied for example for the functions of the form

$$\psi(p) = p^{1/m} L(p), \quad \psi(p) = cp [\ln(p+1)]^r L(\ln p),$$

where $m, c = \text{const} \in (0, \infty)$, $r = \text{const} \in \mathbb{R}$ and $L(\cdot)$ is positive continuous slowly varying at infinity function.

Counterexample. The function

$$\psi_{(r)}(p) = 1, \quad 1 \leq p \leq r;$$

$$\psi_{(r)}(p) = \infty, \quad p > r; \quad r = \text{const} > 1,$$

for which the correspondent function has a form

$$h^*(\ln u) = r \ln u$$

does not satisfy the condition (5.3). Actually, for the r.v. η from the space $L_r = L_r(\Omega)$ the correspondent tail estimate has a form

$$T_\eta(u) \leq cu^{-r},$$

but the inverse conclusion is not true.

6 Concluding remarks.

It is interest by our opinion to obtain the generalization of results of this report into multidimensional case, i.e. into random vectors, alike in the article [7].

We mention even briefly an important possible application of obtained results: a Central Limit Theorem in Banach spaces, in the spirit of [8], section 4.1. The case of finite support $\psi: b < \infty$.

In this case approvals 4.1, 5.1, and 5.2 are in general case incorrect. The correspondent counterexamples may be found in the article [7]. Thus, the problem of description of correspondence between tail behavior and Grand Lebesgue Space norm is in this case an open problem.

Список використаних джерел

1. Булдыгин В.В. Асимптотичні властивості корелограмних оцінок імпульсних перехідних функцій лінійних систем / В.В. Булдыгин, І.П. Блажієвська // Наукові вісті НТУУ "КПІ".– 2010.– № 4.– С. 16–27.
2. Bennet C. Interpolation of operators / C. Bennet, R. Sharpley // Orlando, Academic Press Inc.–1988.
3. Булдыгин В. Метрические характеристики случайных величин и процессов, / В.Булдыгин, Ю.Козаченко.– Киев, ТВиМС.– 1998.
4. Fiorenza A. Duality and reflexivity in grand Lebesgue spaces / A. Fiorenza // Collect. Math.– 2000.– № 51.– P. 131 – 148.
5. Fiorenza A. Grand and small Lebesgue spaces and their analogs / A. Fiorenza and G.E. Karadzhov // Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico, 272/03.–2005.
6. Iwaniec T. On the integrability of the Jacobian under minimal hypotheses / T. Iwaniec and C. Sbordone // Arch. Rat.Mech. Anal., 119.– 1992.– P.129-143.
7. Козаченко Ю.В. Банаховые пространства случайных величин субгауссовского типа / Ю.В. Козаченко, Е.И. Островский // Теор. вер. и Матем. Статист.–1985.– № 32.–С.43–57.
8. Kozachenko Yu.V., Ostrovsky E., Sirota L. Relations between exponential tails, moments and moment generating functions for random variables and vectors / Yu.V.Kozachenko, E.Ostrovsky, L. Sirota // arXiv:1701.01901v1 [math.FA].– 8 Jan 2017.
9. Островский Е.И. Экспоненциальные оценки для случайных полей и их применения / Островский Е.И., // Москва-Обнинск,ОИИПЕ.– 1999.
10. Ostrovsky E. Vector rearrangement invariant Banach spaces of random variables with exponential decreasing tails of distributions / E. Ostrovsky, L. Sirota // arXiv:1510.04182v1 [math.PR].– 14 Oct 2015.
11. Ostrovsky E. Non-asymptotical sharp exponential estimates for maximum distribution of discontinuous random fields /E. Ostrovsky and L. Sirota // arXiv:1510.08945v1 [math.PR].– 30 Oct 2015.
12. Островский Е.И. О вероятностных мерах в сепарабельных пространствах / Островский Е.И.// Совет. Матем. Доклады.– 1980.–№ 255 (6).–С. 836–838.
13. Ostrovsky E. Criterion for convergence almost everywhere, with applications /E. Ostrovsky and L. Sirota // arXiv:1507.04020v1 [math.FA].– 14 Jul 2015.
14. Ostrovsky E. Schlömilch and Bell series for Bessels functions, with probabilistic applications / E. Ostrovsky and L. Sirota // arXiv:0804.0089v1 [math.CV].– 1 Apr 2008.
15. Ostrovsky E. Sharp moment estimates for polynomial martingales /E. Ostrovsky and L. Sirota // arXiv:1410.0739v1 [math.PR].– 3 Oct 2014.
16. Ostrovsky E. Exact exponential bounds for the random field maximum distribution via the majorizing measures (generic chaining) / E. Ostrovsky, E. Rogover // arXiv:0802.0349v1 [math.PR].– 4 Feb 2008.
17. Ostrovsky E. Entropy and Grand Lebesgue Spaces approach for the problem of Prokhorov - Skorokhod continuity of discontinuous random fields. / E. Ostrovsky and L. Sirota // arXiv:1512.01909v1 [math.Pr].– 7 Dec 2015.
18. Ostrovsky E. Fundamental function for Grand Lebesgue Spaces / E. Ostrovsky and L. Sirota // arXiv:1509.03644v1 [math.FA].–11 Sep 2015.
19. Seneta E. Regularly Varying Functions / Seneta E. // Lectures Notes in Mathematics.– 508.–1976.
20. Василик О.І. ϕ – субгауссові випадкові процеси / О.І. Василик, Ю.В. Козаченко, Р.Є.Ямненко// Монографія, Київ, "Київський університет".–2008.

References

1. BENNET C. (1988) *Interpolation of operators*, Orlando, Academic Press Inc.
2. BULDYGIN, V., KOZACHENKO, YU. (2000) *Metric characterization of random variables and random processes*, Amer. Math. Soc., Providence, RI.
3. FIORENZA A. (2000) *Duality and reflexivity in grand Lebesgue spaces*. Collect. Math. **51**, 131 - 148.
4. FIORENZA A AND KARADZHOV G.E. (2005) *Grand and small Lebesgue spaces and their analogs*. Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico 272/03.
5. IWANIEC T. AND SBORDONE C. (1992) *On the integrability of the Jacobian under minimal hypotheses*. Arch. Rat.Mech. Anal., 119, 129-143.
6. KOZACHENKO YU. V., OSTROVSKY E.I. (1985) *The Banach Spaces of random Variables of subgaussian Type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, 32, 43-57.
7. KOZACHENKO YU.V., OSTROVSKY E., SIROTA L. (2017) *Relations between exponential tails, moments and moment generating functions for random variables and vectors*. arXiv:1701.01901v1 [math.FA], 8 Jan 2017.
8. OSTROVSKY E.I. (1999). *Exponential estimations for Random Fields and its applications*, (in Russian), Moscow-Obninsk, OINPE.
9. OSTROVSKY E. AND SIROTA L. (2015) *Vector rearrangement invariant Banach spaces of random variables with exponential decreasing tails of distributions*. arXiv:1510.04182v1 [math.PR], 14 Oct 2015.
10. OSTROVSKY E. AND SIROTA L. (2015) *Non-asymptotical sharp exponential estimates for maximum distribution of discontinuous random fields*. arXiv:1510.08945v1 [math.PR], 30 Oct 2015.
11. OSTROVSKY E.I. (1980) *About supports of probability measures in separable Banach spaces*. Soviet Math., Doklady, V. 255, NO 6, p. 836-838, (in Russian).
12. OSTROVSKY E. AND SIROTA L. (2015) *Criterion for convergence almost everywhere, with applications*. arXiv:1507.04020v1 [math.FA], 14 Jul 2015.
13. OSTROVSKY E. AND SIROTA L. (2008) *Schlömilch and Bell series for Bessels functions, with probabilistic applications*. arXiv:0804.0089v1 [math.CV], 1 Apr 2008.
14. OSTROVSKY E. AND SIROTA L. (2014) *Sharp moment estimates for polynomial martingales*. arXiv:1410.0739v1 [math.PR], 3 Oct 2014.
15. OSTROVSKY E., ROGOVER E. (2008) *Exact exponential bounds for the random field maximum distribution via the majorizing measures (generic chaining)*. arXiv:0802.0349v1 [math.PR], 4 Feb 2008.
16. OSTROVSKY E. AND SIROTA L. (2015) *Entropy and Grand Lebesgue Spaces approach for the problem of Prokhorov - Skorokhod continuity of discontinuous random fields*. arXiv:1512.01909v1 [math.Pr], 7 Dec 2015.
17. OSTROVSKY E. AND SIROTA L. (2015) *Fundamental function for Grand Lebesgue Spaces*. arXiv:1509.03644v1 [math.FA], 11 Sep 2015.
18. SENETA E. (1976) *Regularly Varying Functions*. Lectures Notes in Mathematics, **508**.
19. VASYLYK O.I., KOZACHENKO YU.V., YAMNENKOR.E. (2008) ϕ - *subgaussian random processes*. Monograph, Kyiv, (in Ukrainian).

Надійшла до редколегії: 11.02.2018