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**Про зберігаючі міру гомеоморфізми
просторів шляхів простих стаціонарних
діаграм Браттелі.**

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**On measure preserving
self-homeomorphisms of path spaces of
simple stationary Bratteli diagrams**

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Розглядаються Борелеві міри, які є інваріантними щодо спеціального відношення еквівалентності на просторах шляхів діаграм Браттелі. У роботі вивчається така проблема: Нехай діаграма Браттелі фіксована. Чи кожен гомеоморфізм, який зберігає відповідну міру може бути апроксимований гомеоморфізмами “близькими до фінітарних”? Знайдено певні умови на діаграми для яких відповідь на поставлене питання позитивна.

Ключові слова: Борелева міра, простір шляхів, діаграма Браттелі, група гомеоморфізмів.

Borel measures which are invariant under the tail equivalence relation on path spaces of Bratteli diagrams are considered. We study the following problem: Let Bratteli diagram is fixed. Do every self-homeomorphisms that preserve such a measure can be approximated by homeomorphisms which are “close to finitary” homeomorphisms? We found some conditions on diagrams for which it is true.

Key Words: Borel measure, path space, Bratteli diagram, homeomorphism group.

Статтю представив доктор фізико-математичних наук, професор А.П.Петравчук

1 Introduction

Bratteli diagrams are important objects not only in theory of operator algebras but also in dynamics on Cantor set.

We consider Borel measures μ which are invariant under the tail (cofinal) equivalence relation on path spaces of Bratteli diagrams. It is known the measure μ is unique in the case of a unital simple stationary Bratteli diagram (see [3]). We will consider unital simple stationary Bratteli diagrams only in this work. It is clear that all self-homeomorphisms preserving measure μ form a group. We will denote this group by $\mathcal{M}_B$. Also it is easy to check that the group $\mathcal{M}_B$ is closed in the topology induced by the metrics of $\mathcal{P}(B)$. In this group we distinguish naturally the subgroup $S(B)$ of “finitary” homeomorphisms which consists of all self-homeomorphisms of $\mathcal{P}(B)$ that fix all but finitely many coordinates of paths. The group $S(B)$ has the following properties:

- Elements of $S(B)$ preserve the tail relation.
- $S(B)$ acts transitively on every tail equivalence class.
- $S(B)$ is a direct limit of direct products of

finite symmetric groups.

Therefore the measure is invariant under the tail relation if and only if it is invariant under $S(B)$. Note that $S(B)$ is not closed in the topology induced by the metrics of $\mathcal{P}(B)$. Therefore the measure μ is invariant under the closure $\overline{S(B)}$ of the group $S(B)$.

The following question arises naturally: whether all self-homeomorphisms preserving measure μ can be approximated by “finitary homeomorphisms”?

In the case when $\mathcal{P}(B)$ is the boundary of a spherically homogeneous rooted tree and B is of rank 1 (that is diagram contains only one vertex per level) the group $\mathcal{M}_B$ is the closure of the subgroup $S(B)$ (see [6]). On the other hand there are isometric balls in path-spaces of certain (even) stationary diagram such that they are not permutable by homeomorphisms of $\overline{S(B)}$. But isometric balls have equal measures (Statement 1). Therefore, in this case, it is easily seen that the group $\overline{S(B)}$ is a proper subgroup of the group $\mathcal{M}_B$. It seems naturally to define a subgroup $S(\mathcal{P}(B))$ of the group $\mathcal{M}_B$, slightly bigger than $S(B)$, such that the following requirements hold:

- all isometric balls are permutable (in special

sense) by elements of $S(\mathcal{P}(\mathbf{B}))$;

- if all isometric balls are permutable by elements of $S(\mathbf{B})$, then $S(\mathcal{P}(\mathbf{B}))$ coincides with $S(\mathbf{B})$.

We study the problem whether the group $\mathcal{M}_{\mathbf{B}}$ is the closure of its subgroup $S(\mathcal{P}(\mathbf{B}))$. A unital simple stationary Bratteli diagram is completely defined by its incidence matrix. It is turned out that the group $\mathcal{M}_{\mathbf{B}}$ is the closure of the subgroup $S(\mathcal{P}(\mathbf{B}))$ in the case when the Perron eigenvalue of the incidence matrix is algebraic of maximal possible degree (in fact the degree is equal to the order of the matrix) (Theorem 1).

Some of the results of this paper were announced without proofs in [4].

2 Preliminaries

For definitions of Bratteli diagrams and their path-spaces in general case we refer the reader to [5]. Stationary Bratteli diagrams can be defined in a following way:

A stationary Bratteli diagram $\mathbf{B} = (\{V_i\}_{i \geq 0}, \{E_i\}_{i \geq 1}, \mathbf{s}, \mathbf{r})$ is defined by the following data

- the set of vertices $V = V(\mathbf{B})$ partitioned into a disjoint union $V = \bigsqcup_{i \geq 0} V_i$ of levels, where V_i is a sequence of copies of the same set $V_* = \{v_*^1, \dots, v_*^n\}$ for $i \geq 1$; $V_0 = \{v_0\}$;
- the set of edges $E = E(\mathbf{B})$ also partitioned into a disjoint union $E = \bigsqcup_{i \geq 1} E_i$, where E_i is a sequence of copies of the same set $E_* = \{e_*^1, \dots, e_*^k\}$ for $i \geq 2$; E_1 is a copy of the set V_* ;
- Source and range maps

$$\mathbf{s} : E_i \longrightarrow V_{i-1} \quad ? \quad \mathbf{r} : E_i \longrightarrow V_i$$

act in the same way on each level ($i > 1$), i.e. they are defined by

$$\mathbf{s} : E_* \rightarrow V_* \quad \text{and} \quad \mathbf{r} : E_* \rightarrow V_*.$$

It is also required two additional conditions for the first level:

- the mapping $\mathbf{r} : E_1 \longrightarrow V_1$ is induced by the identity mapping $\text{id} : V_* \rightarrow V_*$,
- $\mathbf{s}(e) = v_0$ for each $e \in E_1$.

Note that we allow more than one edge e starting and ending at the same pair of vertices $\mathbf{s}(e), \mathbf{r}(e)$.

A stationary diagram is called *unital* if for every $v \in V_i$, $i \geq 1$ there exists $e \in E_i$ satisfying $\mathbf{r}(e) = v$.

A stationary diagram $\mathbf{B}$ is called *simple* if for every $v_1, v_2 \in V_*$ there exists a path $(e_1, \dots, e_k)$ in $\mathbf{B}$ such that $\mathbf{s}(e_1) = v_1$ and $\mathbf{r}(e_k) = v_2$.

The matrix $A = (a_{ij})_{i,j=1}^n$ with elements $a_{ij} = |\{e \mid e \in E_*, \mathbf{s}(e) = j, \mathbf{r}(e) = i\}|$ is called the *incidence matrix* of a stationary diagram $\mathbf{B}$.

The space of paths $\mathcal{P}(\mathbf{B})$ of a stationary Bratteli diagram is the set of sequences $\{(e_1, e_2, e_3, \dots) \mid e_i \in E_i, \mathbf{r}(e_i) = \mathbf{s}(e_{i+1}), i \geq 1\}$. Thus, it is equal to the set of all sequences $(e_1, e_2, e_3, \dots) \in V_* \times E_*^{\mathbb{N}}$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for all i , $i \geq 1$.

We define spherically homogeneous rooted tree T which levels $V_m(T)$, $m \geq 1$ consist of $V_* \times E_*^{m-1}$, $V_0(T) = \{\emptyset\}$. Two vertices are adjacent in the tree T if they are contained in the consecutive levels and if they are beginnings of the same sequence of $V_* \times E_*^{\mathbb{N}}$. In this tree we distinguish the subtree $T(\mathbf{B})$ which contains vertex $v \in V_m(T)$ if and only if v is a beginning of a sequence of $\mathcal{P}(\mathbf{B})$. The path space $\mathcal{P}(\mathbf{B})$ can be identified with the boundary $\partial T(\mathbf{B})$ of the tree $T(\mathbf{B})$.

A set of paths of $\partial T(\mathbf{B})$ going through a vertex v is called *ball* or *cylinder set corresponding to v* . We will say that a ball U_v ($v \in V_m$) is a ball of the level V_m . Note that the cylinder set corresponding to a finite path $\bar{e} = (e_1, \dots, e_m)$ in $\mathbf{B}$ is the set $[\bar{e}] = \{x = (x_i) \in \mathcal{P}(\mathbf{B}) \mid x_i = e_i, 1 \leq i \leq m\}$. All cylinder sets of $\mathcal{P}(\mathbf{B})$ form a basis of open sets of the topological space $\mathcal{P}(\mathbf{B})$. This topology is induced by the ultrametrics ρ given by $\rho(\bar{e}, \bar{e}') = 2^{-m}$ if $\bar{e} \neq \bar{e}'$, where m is the length of common beginning of the sequences $\bar{e}$ and $\bar{e}'$, and $\rho(\bar{e}, \bar{e}') = 0$ if $\bar{e} = \bar{e}'$. The set of the cylinder sets of $\mathcal{P}(\mathbf{B})$ are exactly the set of all balls of the metric space $(\mathcal{P}(\mathbf{B}), \rho)$. Note that if $\mathbf{r}(e_m) = \mathbf{r}(e'_m)$, then the balls $[\bar{e}]$ and $[\bar{e}']$ are isometric.

Two paths $\bar{e}$ and $\bar{e}'$ of $\mathcal{P}(\mathbf{B})$ are *cofinal* if $e_m = e'_m$ for all m sufficiently large. It is easily seen that the cofinal relation is an equivalence relation.

A finite Borel measure μ on $\mathcal{P}(\mathbf{B})$ is *invariant* if it is invariant with respect to the cofinal equivalence. We assume that μ takes finite values on some clopen sets.

A unital simple stationary diagram B admits a unique invariant probability measure μ (see [3]). Due to [1] the following equality holds $\mu([\bar{e}]) = \mu([\bar{e}'])$ for arbitrary paths $\bar{e} = (e_1, \dots, e_m)$ and $\bar{e}' = (e'_1, \dots, e'_m)$ satisfying $r(e_m) = r(e'_m)$.

The group $S_m(B)$ is, by definition, the group of all self-homeomorphisms of $\partial T(B)$, that just permute the balls corresponding to vertices of the level V_m , that is, $S_m(B)$ is the group of all self-homeomorphisms of $\partial T(B)$ that fix coordinates of paths starting with $(m+1)$ th coordinate.

Clearly, $S_m(B) < S_{m+1}(B)$. The group $S(B)$ of “finitary” homeomorphisms is a union of the ascending chain of subgroups $S_m(B)$ of the group of all self-homeomorphisms of $\partial T(B)$ (also this subgroup is called the full AF-group of B [2]).

On the group $\text{Homeo}(\mathcal{P}(B))$ of all self-homeomorphisms of $\mathcal{P}(B)$ we will consider the metrics $\bar{\rho}$, which is induced by the metrics ρ of the space $\mathcal{P}(B)$: $\bar{\rho}(g, h) = \max_{x \in \partial T} \rho_{\bar{\lambda}}(x^g, x^h)$ for all $g, h \in \text{Homeo}(\mathcal{P}(B))$.

3 Results

Let us fix and recall our notations. An unital simple stationary Bratteli diagram is denoted by B . The path space of B is denoted by $\mathcal{P}(B)$. The rooted tree corresponding to B is denoted by $T(B)$, $\partial T(B) = \mathcal{P}(B)$. The ball corresponding to the vertex $v \in V(T(B))$ is denoted by U_v . Matrix $A = (a_{ij})_{i,j=1}^n$ is the incidence matrix of B . Let M^T denote the transpose of a matrix M . The Perron eigenvalue of A will be denoted by λ_A , that is, λ_A is the largest real eigenvalue of the matrices A and A^T . The vector $v_\lambda = (v_1, v_2, \dots, v_n)$ with positive coordinates is the eigenvector of A^T corresponding to λ_A . Additionally we assume that $v_1 + v_2 + \dots + v_n = 1$. The probabilistic Borel measure on $\mathcal{P}(B)$ which is invariant under the tail relation is denoted by μ .

It is known how to compute the measure of a cylinder set $[(\bar{e})]$ ($(\bar{e}) = (e_1, \dots, e_m)$) of the path space $\mathcal{P}(B)$: $\mu([\bar{e}]) = \lambda^{-m} v_i$ where i is determined by the equality $r(e_m) = v_i^*$.

Now we want to define the subgroup $S(\mathcal{P}(B))$ of the group $\text{Homeo} \mathcal{P}(B)$.

At first we define the notion of permutation of isometric balls. We can certainly assume that the set E_1 coincides with V_* . Let $\bar{v}_*^i = (v_*^i)$ where $1 \leq i \leq n$. We will denote by $e \sim e'$ a pair of paths $\bar{e} = (e_1, \dots, e_m)$ and $\bar{e}' = (e'_1, \dots, e'_m)$ such

that corresponding cylinder sets $[\bar{e}]$ and $[\bar{e}']$ are isometric. For v_*^i, v_*^j satisfying $v_*^i \sim v_*^j$, $i < j$ and $v_*^i = r(v_*^i) \neq r(v_*^j) = v_*^j$ there exists a bijection

$$\psi_{ij} : \{b \mid b \in E_*, s(b) = v_*^i\} \rightarrow \{c \mid c \in E_*, s(c) = v_*^j\},$$

such that $\psi_{ij}(b) = c$ implies $b \sim c$. Let us fix this bijection. Let us fix $\psi_{ji} = \psi_{ij}^{-1}$ too.

Let paths $\bar{e} = (e_1, \dots, e_m)$ and $\bar{e}' = (e'_1, \dots, e'_m)$ be such that $e \sim e'$. We will define a mapping $\phi_{ee'} : [\bar{e}] \rightarrow [\bar{e}']$. If $r(e_m) = r(e'_m)$ then we put $\phi_{ee'}(e_1, \dots, e_m, x_1, x_2, \dots) = (e'_1, \dots, e'_m, x_1, x_2, \dots)$, for all possible paths of $\mathcal{P}(B)$ of the form $(e_1, \dots, e_m, x_1, x_2, \dots)$. If $r(e_m) \neq r(e'_m)$, then $\phi_{ee'}(e_1, \dots, e_m, x_1, x_2, \dots)$ is determined by induction. Let $r(e_m) = v_*^i$ and $r(e'_m) = v_*^j$. Let $y_1 = \psi_{ij}(x_1)$. If $r(x_1) = r(y_1)$, then $\phi_{ee'} : [\bar{x}_1] \rightarrow [\bar{y}_1]$ was already defined, and $\phi_{ee'}(e_1, \dots, e_m, x_1, x_2, x_3, \dots) = (e'_1, \dots, e'_m, y_1, x_2, x_3, \dots)$. Otherwise, we go to the consequent element of the sequence. Set $y_2 = \psi_{kl}(x_2)$, where k and l are determined from the following equalities $r(x_1) = v_*^k$ and $r(y_1) = v_*^l$. If $r(x_2) = r(y_2)$, then $\phi_{ij}(e_*^i, x_1, x_2, x_3, \dots) = (e_*^j, y_1, y_2, x_3, \dots)$. Otherwise, we go to the consequent element of the sequence, and so on.

We call the mapping $\phi_{ee'}$ the permutation of isometric balls $[\bar{e}]$ and $[\bar{e}']$. It is easily seen that $\phi_{ee'}$ induces an isometry of rooted trees $T_{\bar{e}}(B) \rightarrow T_{\bar{e}'}(B)$. Since every isometry of rooted trees induce an isometry of corresponding boundaries $\phi_{ee'}$ is an isometry.

Thus we can define the group $S_m(\mathcal{P}(B))$, which consists of all self-homeomorphisms of $\mathcal{P}(B)$ that just permute balls of the level V_m . By the construction the groups $S_m(\mathcal{P}(B))$ ($m \in \mathbb{N}$) form an ascending chain of subgroups of $\text{Homeo} \mathcal{P}(B)$. Therefore, it is natural to define in the group $\text{Homeo} \mathcal{P}(B)$ the subgroup $S(\mathcal{P}(B))$, which is the union of ascending chain of subgroups $S_m(\mathcal{P}(B))$ ($m \in \mathbb{N}$).

Note that if for a diagram B from the fact that balls $[\bar{e}]$ and $[\bar{e}']$ are isomorphic it follows that $r(e_m) = r(e'_m)$, then $S(\mathcal{P}(B))$ coincides with $S(B)$.

Proposition 1. The group $S(\mathcal{P}(B))$ is a subgroup of $\mathcal{M}_B$.

Proof. In fact, we only need to prove that isometric balls of the first level have equal measures. Namely, we need to prove that the equality $v_l = v_m$ follows from the equality $v_*^l \sim$

v_*^m . Since the relation $\sim$ is induced by the relation of isometry of balls then $\sim$ is an equivalence relation.

The set V_* is partitioned into classes of the equivalence $\sim$: $V_*/\sim = \{K_1, K_2, \dots, K_s\}$ for some $s \geq 1$. Without loss of generality we can assume that this partition satisfies equalities $K_i = \bigsqcup_{j=k_i}^{k_{i+1}-1} v_*^j$ for $i = 1, \dots, s$ and for some $k_1, \dots, k_{s+1}$ where $1 = k_1 < k_2 < \dots < k_{s+1} = n$.

For any pair of isometric balls U_1 and U_2 an isometry $U_1 \rightarrow U_2$ induces bijections between certain balls of the first level of U_1 and U_2 . In terms of elements of the incidence matrix it may be written as

$$\sum_{p=k_i}^{k_{i+1}-1} a_{pq} = \sum_{p=k_i}^{k_{i+1}-1} a_{pt}, \quad (1)$$

for $k_j \leq q, t \leq k_{j+1} - 1$ and for all $i = 1, \dots, s + 1$ and $j = 1, \dots, s$.

We will consider the matrix $A' = (a'_{ij})_{i,j=1}^s$ obtained from A as follows: $a'_{ij} = \sum_{p=k_i}^{k_{i+1}-1} a_{pq}$, where q satisfies the inequality $k_j \leq q \leq k_{j+1} - 1$. Due to the equality (1) the matrix A' is well defined. Let $v' = (v'_1, \dots, v'_s)$ be the Perron eigenvector of the matrix $(A')^T$. Then the vector

$$v'' = (\underbrace{v'_1, \dots, v'_1}_{k_2-k_1}, \underbrace{v'_2, \dots, v'_2}_{k_3-k_2}, \dots, \underbrace{v'_s, \dots, v'_s}_{k_{s+1}-k_s})$$

is an eigenvector of the matrix A^T . It has positive coordinates by construction. Since positive eigenvector of positive irreducible matrix is unique, it follows that v'' is the Perron eigenvector of the matrix $(A')^T$. Thus v'' is collinear to v . Therefore, $v_*^l \sim v_*^m$ implies $v_l = v_m$, which completes the proof. $\square$

The procedure of constructing the matrix A' from A described in the proof of Proposition 1 we shall call *the reduction*. First we note that all coordinates of the Perron eigenvector of A'^T are pairwise different.

Note also that the closure of the group $S(\mathcal{P}(\mathcal{B}))$ contains the group of all bijective local self-isometries of the path space $\mathcal{P}(\mathcal{B})$. Namely, we have proved that the group of all bijective local self-isometries of the path space $\mathcal{P}(\mathcal{B})$ is a subgroup of $\mathcal{M}_{\mathcal{B}}$.

Lemma 1. The group $\mathcal{M}_{\mathcal{B}}$ is closed in the topology induced by the metrics $\bar{\rho}$.

Proof. We will prove that for a ball B and for a sequence of measure-preserving homeomorphisms $\{f_n\}$ converging to f the following equality holds $\mu(f(B)) = \mu(B)$. Since $\{f_n\}$ converges to f , for any positive ϵ there exists N such that for every $n > N$ the ϵ -neighborhood of $f(B)$ contains $f_n(B)$. The image $f(B)$ is a finite union of balls. Therefore $\mu(f(B)) + \delta \geq \mu(B)$ for any $\delta > 0$. Thus $\mu(f(B)) \geq \mu(B)$. We can decompose $\mathcal{P}(\mathcal{B})$ into finite disjoint union of balls provide this union contains given ball. We also have $\mu(f(\mathcal{P}(\mathcal{B}))) = \mu(\mathcal{P}(\mathcal{B}))$. It follows that $\mu(f(B)) = \mu(B)$ for any ball B of $\mathcal{P}(\mathcal{B})$.

Since the set of all balls forms the base of the topology induced by the metrics $\bar{\rho}$, the homeomorphism f preserves the measure μ . $\square$

Proposition 2. Let $\mathcal{B}$ be a simple stationary unital Bratteli diagram, A be the incidence matrix of $\mathcal{B}$, A' be the matrix obtained by reduction from the matrix A , $\mathcal{B}'$ be the Bratteli diagram defined by A' . Then $\mathcal{M}_{\mathcal{B}}$ is the closure of $\text{LIsom}(\mathcal{B})$ if and only if $\mathcal{M}_{\mathcal{B}'}$ is the closure of $S(\mathcal{B}')$.

Proof. It is easily seen that reduction (ambiguously) defines an embedding of the space $\mathcal{P}(\mathcal{B}')$ into $\mathcal{P}(\mathcal{B})$, and this embedding induces an embedding of $\mathcal{M}_{\mathcal{B}'}$ into $\mathcal{M}_{\mathcal{B}}$. By the other hand for any large enough level number m in $\mathcal{P}(\mathcal{B}')$ there exists a union U of certain balls of the level number m with the following property: the largest subgroup of $\mathcal{M}_{\mathcal{B}'}$ that consists of all elements acting trivially on the complement to U is isomorphic to $\mathcal{M}_{\mathcal{B}}$. Therefore, there exists an element of $\mathcal{M}_{\mathcal{B}}$ which is approximated by elements of $\text{LIsom}(\mathcal{B})$ if and only if there exists an element of $\mathcal{M}_{\mathcal{B}'}$ which is approximated by elements of $S(\mathcal{B}')$. $\square$

Due to Proposition 2 we will consider Bratteli diagrams with only

Lemma 2. Let $\mathcal{B}$ be such that all coordinates of v_{λ} are pairwise different. Assume that for any finite unions of balls $U, W \subset \mathcal{P}(\mathcal{B})$ the equality $\mu(U) = \mu(W)$ implies that there is $h \in S(\mathcal{B})$ satisfying $h(U) = W$. Then there is a convergent to g sequence $\{h_n \in S(\mathcal{B}), n \in \mathbb{N}\}$ for any $g \in \mathcal{M}_{\mathcal{B}}$. Thus in this case the group $\mathcal{M}_{\mathcal{B}}$ is the closure of its subgroup $S(\mathcal{B})$.

Proof. Assume that the equality $\mu(U) = \mu(W)$ implies that there exists $h \in S(\mathcal{B})$ such that $h(U) = W$ for a finite unions of balls $U, W \subset$

$\mathcal{P}(\mathbf{B})$. Fix $g \in \mathcal{M}_{\mathbf{B}}$. We will construct a required sequence $\{h_n \in S(\mathcal{P}(\mathbf{B})), n \in \mathbb{N}\}$ by induction. At first we will choose h_1 . Let V_{k_1} be some level of the tree $T(\mathbf{B})$. For every ball U_{v_i} ($v_i \in V_{k_1}$) there is $m_1(v_i) \in \mathbb{N}$ such that the image $g^{-1}(U_{v_i})$ is the disjoint union of balls $g^{-1}(U_{v_i}) = \bigsqcup_{w_i \in W} U_{w_j}$ for some $W \subset V_{m_1(v_i)}$. Since g preserves the measure μ , by assumption, there exists $h_* \in S(\mathbf{B})$ such that $h_*(U_{v_i}) = g^{-1}(U_{v_i})$. Let $l_1(v_i) \in \mathbb{N}$ such that $h_* \in S_{l_1(v_i)}(\mathbf{B})$. Set $n_1 = 1 + \max\{m_1(v_i), l_1(v_i), k_1 \mid v_i \in V_{k_1}\}$. Note that $n_1 > k_1$. Hence there is $h_1 \in S_{n_1}(\mathbf{B})$ such that $g^{-1}(U_{v_i}) = h_1^{-1}(U_{v_i})$ for any $v_i \in V_{k_1}$. Therefore for every $x \in \mathcal{P}(\mathbf{B})$ there is $v \in V_{k_1}$ such that $g(x) \in U_v$ and $h_1(x) \in U_v$. Thus $\bar{\rho}(g, h_1) < 2^{-k_1}$.

Choose h_{i+1} by induction. Assume that h_i , k_i , and n_i are already defined. Putting $k_{i+1} = n_i$ for $i \geq 1$ we define h_{i+1} in a similar way. Thus $\bar{\rho}(g, h_{i+1}) < 2^{-k_{i+1}}$. By construction, the obtained sequence $\{k_1, k_2, \dots\}$ is strictly increasing. Hence the sequence $\{h_n \in S(\mathcal{P}(\mathbf{B})), n \in \mathbb{N}\}$ converges to g as required. $\square$

Let us consider a diagram $\mathbf{B}$ with the incidence matrix A . Assume that all coordinates of the Perron eigenvector of the matrix A^T are pairwise different. Since coordinates of v_λ are pairwise different there are exactly n values

$$\{p_{i,m} = \lambda^{-m} v_i \mid i = 1, \dots, n\}$$

of the measure μ for balls of the level m . We call balls of the measure $p_{i,m}$ the balls of type i ($1 \leq i \leq n$). Note that $S_m(\mathbf{B})$ acts naturally on the set V_m and is a direct product of symmetrical groups each of which acts naturally on the vertices of fixed type.

Proposition 3. Let $\mathbf{B}$ be a simple stationary unital Bratteli diagram of rank n and A be the incidence matrix of $\mathbf{B}$. Assume that all coordinates of the Perron eigenvector $v_\lambda = (v_1, \dots, v_n)$ of the matrix A^T are linearly independent. Then the group $\mathcal{M}_{\mathbf{B}}$ is the closure of its subgroup $\text{LIso}(\mathbf{B})$.

Proof. Since all coordinates of v_λ are different we have $\overline{\text{LIso}(\mathbf{B})} = \overline{S(\mathbf{B})}$. Therefore by Lemma 2 it is sufficient to show that for any finite unions of balls $U, W \subset \mathcal{P}(\mathbf{B})$ the equality $\mu(U) = \mu(W)$ implies that there exists $h \in S(\mathbf{B})$ satisfying $h(U) = W$.

Let $U, W \subset \mathcal{P}(\mathbf{B})$ and $\mu(U) = \mu(W)$. We can assume that U and W are unions of balls of the

level V_m . Let k_i denote the number of balls contained in U of the measure $p_{i,m}$ ($1 \leq i \leq n$). Let l_i denote the number of balls contained in W of the measure $p_{i,m}$ ($1 \leq i \leq n$). Since $\mu(U) = \mu(W)$, we have $\sum_{i=1}^n k_i v_i = \sum_{i=1}^n l_i v_i$. Hence

$$\sum_{i=1}^n (k_i - l_i) v_i = 0. \quad (2)$$

Since $v_1, \dots, v_n$ are linearly independent we have $k_i = l_i$ for $i = 1, \dots, n$. $\square$

Theorem 1. Let $\mathbf{B}$ be a simple stationary unital Bratteli diagram of rank n and A be the incidence matrix of $\mathbf{B}$. Assume that one of the following conditions holds

- 1) The characteristic polynomial of the matrix A is irreducible.
- 2) The determinant of A is equal to 0 and $n = 2$.

Then the group $\mathcal{M}_{\mathbf{B}}$ is the closure of its subgroup $\text{LIso}(\mathbf{B})$.

Proof. Due to Proposition 2 there is no loss of generality in assuming that all coordinates of the Perron eigenvector of the matrix A^T are pairwise different. In this case $\overline{\text{LIso}(\mathbf{B})} = \overline{S(\mathbf{B})}$. Therefore by Lemma 2 it is sufficient to show that for any finite unions of balls $U, W \subset \mathcal{P}(\mathbf{B})$ the equality $\mu(U) = \mu(W)$ implies that there exists $h \in S(\mathbf{B})$ satisfying $h(U) = W$.

Let $U, W \subset \mathcal{P}(\mathbf{B})$ and $\mu(U) = \mu(W)$. We can assume that U and W are unions of balls of the level V_m . Let k_i denote the number of balls contained in U of the measure $p_{i,m}$ ($1 \leq i \leq n$). Let l_i denote the number of balls contained in W of the measure $p_{i,m}$ ($1 \leq i \leq n$). Since $\mu(U) = \mu(W)$, we have $\sum_{i=1}^n k_i v_i = \sum_{i=1}^n l_i v_i$. Hence

$$\sum_{i=1}^n (k_i - l_i) v_i = 0. \quad (3)$$

Consider the theorem conditions.

1. The first condition implies that λ_A is an algebraic number of degree n . If $n = 1$, then $T(\mathbf{B})$ is a spherically homogeneous tree. In this case all balls of any level V_m are pairwise isometric which implies the assertion (see also [6]). Fix $n > 1$. Let us consider the matrix $A - \lambda_A I$ where I is the identity matrix. Suppose that $k_i - l_i \neq 0$ for some i . We substitute i th row of the matrix $A - \lambda_A I$ by the vector $w = (k_1 - l_1, \dots, k_n - l_n)$. The determinant of the obtained matrix is equal

to 0 due to (3). The coordinates of the vector w are integers. Therefore λ_A is a root of polynomial with integer coefficients of degree $n - 1$. Hence the characteristic polynomial of the matrix A is reducible. This contradicts with our assumption. Thus, $k_i = l_i$ for $1 \leq i \leq n$. Therefore, there exists $h \in S_m(\mathbf{B})$ such that $h(U) = W$.

2. Now let the determinant of A be equal to 0 and $n = 2$. We will show that the required h can be taken from $S_{m+1}(\mathbf{B})$. For this purpose it is sufficient to prove that the sets U and V contain the same number of balls of type i on the level V_{m+1} for $i = 1, 2$.

Since the determinant of A is equal to 0, we have $\lambda_A = a_{11} + a_{22}$ and there exists a non-zero vector with integer coordinates $w = (w_1, w_2)$ sat-

isfying $Aw^T = 0$. Therefore, $w_1v_1 + w_2v_2 = 0$. We can assume that w_1 and w_2 are coprime. Applying (3) we obtain $(k_1 - l_1, k_2 - l_2) = aw$ where a is an integer. Thus we only need to show that $w_1(a_{11} + a_{21}) = -w_2(a_{12} + a_{22})$ (i.e. the sets U and V contain the same number of balls of type i on the level V_{m+1} for $i = 1, 2$). But the last equality is a consequence of the equality $Aw^T = 0$. Therefore, h can be taken from $S_{m+1}(\mathbf{B})$, which completes the proof. $\square$

It is easy to check that an equivalent formulation of the first condition of the last theorem is: the coordinates of the Perron eigenvector of the matrix A^T are linearly independent over $\mathbb{Z}$.

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