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Steady-state sloshing in an orbitally-forced square-base tank

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Ця стаття є продовженням серії теоретичних робіт Фалтінсена та Тимохи, присвячених резонуючим усталеним хлюпанням рідини в твердому баці з квадратним перерізом, із використанням нелінійної модальної системи Наріманова-Мойсеева, яка вперше була виведена та застосована цими вченими для дослідження проблем хлюпання рідини. Системи рівнянь такого виду можливо застосувати до коливальних, поступальних, хвильових та кренових періодичних збурень, але, через складність математичного апарату, попередні роботи досліджували лише частинні випадки цих рухів, а саме зворотно-поступальні (повздожні, поперечні та діагональні) у різних типах баків. В цій роботі показано, що усталені хвилі, під дією таких типів збурення, асимптотично ідентичні до таких, які виникають внаслідок горизонтального орбітального збурення в баку.

Ключові слова: хлюпання рідини, демпфування, усталені хвилі.

The paper conducts a series of analytical studies on the resonant steady-state sloshing in a rigid square base container, which have been originated by Faltinsen & Timokha who derived and applied the Narimanov-Moiseev-type nonlinear modal equations for investigation in the sloshing problem. The modal equations, which consist on nine-dimensional system of ordinary differential equations, should be applicable for sway/pitch/surge/roll periodic excitations but, due to, basically, mathematical difficulties, the previous papers exclusively concentrated on the reciprocating (longitudinal, oblique and diagonal) motions of the container. This article is showed that the steady-state waves caused by this kind of forcing are asymptotically identical to those occurring when the tank performs horizontal orbital motions. We generalize the previous results by Faltinsen & Timokha to classify the steady-state wave regimes versus the semi-axes ratio of the forcing ellipse in the tank which is filled by a liquid with a finite depth.

Key Words: sloshing, damping, steady-state waves.

Статтю представив д.ф.-м.н., проф. Жук Я.О.

A rigid square base rigid tank is partially filled by a perfect incompressible (irrotational flows) liquid with a finite liquid depth. The tank moves periodically by surge, sway, roll, and pitch with the prescribed translatory angular motions as shown in fig. 1. Sloshing is considered in the non-inertial tank-fixed coordinate system.

The free surface $\Sigma(t): z = f(x, y, t)$ and the absolute velocity potential $\Phi(x, y, z, t)$ should simultaneously be found from the corresponding free-surface problem or its variational analogy. When focusing on the steady-state wave motions in the tank, one should find a periodic solution of the problem, namely $T = 2\pi / \sigma$ -periodic (σ is the forcing frequency) function $f(x, y, t)$ and $\Phi(x, y, z, t)$. Following [1],

getting the required periodic solutions may employ the nine-dimensional system of nonlinear ordinary differential (modal) equations, which are based on the Narimanov-Moiseev approximation of the resonant liquid sloshing. The asymptotic approximation suggests the small (relative to tank breadth L) non-dimensional forcing amplitude

$$\eta_i(t) \sim O(\epsilon) \quad (1)$$

where $\eta_i(t)$ are the generalised coordinates determining the container motions with four degrees of freedom.

Fig. 1. The sketch of the spatially-moving tank with four degrees of freedom, which explains which the generalised coordinates in (1)

In addition, we consider the L -scaled Fourier (modal) representation of the free surface

$$\begin{aligned} f(x, y, t) = & a_1(t)f_1(x) + b_1(t)f_1(y) + \\ & a_2(t)f_2(x) + b_2(t)f_2(y) + c_1(t)f_1(x)f_1(y) + \\ & a_3(t)f_3(x) + b_3(t)f_3(y) + c_{21}(t)f_2(x)f_1(y) + \\ & c_{12}(t)f_2(y)f_1(x) + [linear] + o(m) \end{aligned} \quad (2)$$

where the first two terms have the $o(m^{1/3})$ order, the next three terms are of the order $o(m^{2/3})$ and the remaining quantities, without linear components, have $o(m)$ order. Here the spatial components $f_i^{(1)}(x)f_j^{(2)}(y)$ are the natural sloshing modes determined by the two-dimensional Stokes waves

$$f_i^{(1)}(x) = \cos(\pi i(x + 1/2)),$$

$$f_i^{(2)}(y) = \cos(\pi i(y + 1/2)), i \geq 0$$

According to [1], the lowest-order sloshing-related generalised coordinates in (2) are governed by the Narimanov-Moiseev-type nonlinear modal equations

$$\begin{aligned} & \ddot{a}_1 + 2\xi_{1,0}\sigma_{1,0}\dot{a}_1 + \sigma_{1,0}^2 a_1 + d_1 \left(\ddot{a}_1 a_2 + \dot{a}_1 \dot{a}_2 \right) + \\ & + d_2 \left(\ddot{a}_1 a_1^2 + \dot{a}_1^2 a_1 \right) + d_3 \ddot{a}_2 a_1 + d_6 \ddot{a}_1 b_1^2 + \\ & + b_1 (d_7 c_1 + d_8 a_1 b_1 + d_9 c_1 b_1 + d_{10} \dot{b}_1^2 a_1 + \\ & + d_{11} \dot{a}_1 \dot{b}_1 b_1 + d_{12} \dot{b}_1 \dot{c}_1) = -P_{1,0} \left(\ddot{\eta}_1 - S_{1,0} \ddot{\eta}_5 - g \eta_5 \right) = K_x(t), \\ & \ddot{b}_1 + 2\xi_{0,1}\sigma_{0,1}\dot{a}_1 + \sigma_{0,1}^2 b_1 + d_1 \left(\ddot{b}_1 b_2 + \dot{b}_1 \dot{b}_2 \right) + \\ & + d_2 \left(\ddot{b}_1 b_1^2 + \dot{b}_1^2 b_1 \right) + d_3 \ddot{b}_2 b_1 + d_6 \ddot{b}_1 a_1^2 + \ddot{a}_1 (d_7 c_1 + d_8 a_1 b_1) \\ & + d_9 c_1 a_1 + d_{10} d_{11} \dot{a}_1 \dot{b}_1 a_1 + d_{12} \dot{a}_1 \dot{c}_1 = -P_{0,1} \left(\ddot{\eta}_2 + S_{0,1} \ddot{\eta}_4 + g \eta_4 \right) = K_y(t); \end{aligned}$$

where the natural (circular) sloshing frequencies

$$\sigma_{j,i} = \sigma_{i,j} = \sqrt{g\pi \sqrt{i^2 + j^2} \tanh(\pi \sqrt{i^2 + j^2} h)}$$

$$P_{i,0} = P_{0,i} = \frac{2}{\pi i} \tanh(\pi i h) [(-1)^i - 1],$$

$$S_{i,0} = S_{0,i} = \frac{2}{\pi i} \tanh(\pi i h / 2).$$

The hydrodynamic coefficients at the nonlinear terms are functions of the mean liquid depth h .

When constructing the asymptotic periodical solution, we introduce the lowest-order component of the periodic solution

$$a_1(t) = a \cos \sigma t + \bar{a} \sin \sigma t + O(m);$$

$$b_1(t) = \bar{a} \cos \sigma t + a \sin \sigma t + O(m),$$

Substitute them in the nonlinear modal equations, gather the similar quantities at different Fourier harmonics and show that the lowest-order amplitudes are governed by the secular equations

$$\begin{cases} a \left[\Lambda + m_1 \left(a^2 + \bar{a}^2 \right) + m_2 \bar{b}^2 + m_3 b^2 \right] + \bar{a} \left[(m_2 - m_3) \bar{b} b + \xi \right] = m_x, \\ \bar{a} \left[\Lambda + m_1 \left(a^2 + \bar{a}^2 \right) + m_2 b^2 + m_3 \bar{b}^2 \right] + a \left[(m_2 - m_3) \bar{b} b - \xi \right] = \bar{m}_x, \\ b \left[\Lambda + m_1 \left(b^2 + \bar{b}^2 \right) + m_2 \bar{a}^2 + m_3 a^2 \right] + \bar{b} \left[(m_2 - m_3) \bar{a} a - \xi \right] = m_y, \\ \bar{b} \left[\Lambda + m_1 \left(b^2 + \bar{b}^2 \right) + m_2 a^2 + m_3 \bar{a}^2 \right] + b \left[(m_2 - m_3) \bar{a} a + \xi \right] = \bar{m}_y \end{cases}$$

where

$$a^2 + \bar{a}^2 = aa, \quad b^2 + \bar{b}^2 = bb, \quad m_2 - m_3 = m_{23}.$$

This means that $a, \bar{a}, b, \bar{b} = O(m^{1/3})$ become functions of the forcing frequency $\Lambda = 1 - \sigma_1^2 / \sigma^2$, the damping coefficient $\xi = 2\xi_{0,1} = 2\xi_{1,0}$ and the $O(m)$ -order non-dimensional forcing amplitudes $m_x, \bar{m}_x, m_y$ and $\bar{m}_y$ in the front of the lowest Fourier harmonics in the right-hand sides of the modal equations ($K_x(t)$ and $K_y(t)$), i.e.

$$m_x = \frac{\sigma}{\pi} \int_0^{2\pi/\sigma} \cos \sigma t K_x(t) dt;$$

$$\bar{m}_x = \frac{\sigma}{\pi} \int_0^{2\pi/\sigma} \sin \sigma t K_x(t) dt,$$

$$\bar{m}_y = \frac{\sigma}{\pi} \int_0^{2\pi/\sigma} \cos \sigma t K_y(t) dt;$$

$$m_y = \frac{\sigma}{\pi} \int_0^{2\pi/\sigma} \sin \sigma t K_y(t) dt.$$

Because we consider sloshing due to the primary resonance, i.e., the forcing frequency is close to the lowest natural sloshing frequency, the right-hand side of the secular equations is not zero, i.e., at least, one of the non-dimensional amplitude parameters $m_x, \bar{m}_x, \bar{m}_y$ and m_y is not be zero. Based on the input

(forcing) amplitude parameters $m_x, \bar{m}_x, \bar{m}_y$ and m_y , one can identify the types of the tank forcing in terms of the primary resonance but adopting the secular system solution, $a, \bar{a}, b, \bar{b}$, as a function of σ / σ_1 , we can specify (classify) the types of liquid sloshing regimes and analyse the stability of these regimes.

When analysing the Narimanov-Moiseev-type approximation of the steady-state resonant sloshing in a circular base tank, [2] showed that each steady-state wave regime caused by the sway/surge/pitch/roll

periodic tank excitations has, to within the $O(m)$ contribution (which does not affect the wave type and its stability), an equivalent steady-state wave regime excited by a suitable orbital horizontal (elliptic) container movement. The same holds true for the square base container. Indeed, the non-dimensional

wave amplitude parameters $a, \bar{a}, b, \bar{b}$ are exclusively linked in with the four non-dimensional non-zero amplitudes $m_x, \bar{m}_x, \bar{m}_y$ and m_y associated with the lowest Fourier harmonics of the periodic tank forcing $\eta_i(t), i=1,2,4,5$. The higher harmonics do not

influence $a, \bar{a}, b, \bar{b}$ and, as we remarked in previous sections, the first $O(m^{1/3})$ and second $O(m^{2/3})$ components of the asymptotic periodic solution and its stability. The same non-dimensional forcing amplitudes $m_x, \bar{m}_x, \bar{m}_y$ appear for the orbital (cyclic) horizontal tank motions by

$$\eta_1(t) = e_x \cos \sigma t + \bar{e}_x \sin \sigma t,$$

$$\eta_2(t) = \bar{e}_y \cos \sigma t + e_y \sin \sigma t,$$

$$\eta_4(t) = \eta_5(t) \equiv 0,$$

$$m_x = P_1 e_x, \bar{m}_x = P_1 \bar{e}_x, \bar{m}_y = P_1 \bar{e}_y, m_y = P_1 e_y; P_1 = P_{1,0} = P_{0,1} < 0$$

so that, when

$$e_x e_y - \bar{e}_x \bar{e}_y = 0, \quad (4)$$

the tank trajectory degenerates to a straight line, namely, it reduces to the reciprocating tank movements.

This means that the types of the steady-state resonant wave regimes and their stability for the periodic sway/pitch/roll/surge tank movements can, within the framework of the Narimanov-Moiseev approximation, be associated with asymptotically equivalent steady-state wave regimes caused

by $m_x, \bar{m}_x, \bar{m}_y$ and m_y . Using an appropriate time lag

$t := t + t_0$, one can exclude $\bar{e}_x$ and focus on the only three input amplitude parameters.

Fig. 2. The three types of equivalent orbital forcing caused.

When (4) is not satisfied, the orbital horizontal tank forcing determines a *centred elliptic trajectory* (of the tank) in the O_{xy} -plane, which is governed by

$$\left(e_y^2 + \bar{e}_y^2\right)x^2 + \left(e_x^2 + \bar{e}_x^2\right)y^2 - 2\left(e_x \bar{e}_y + \bar{e}_x e_y\right)xy = (e_x e_y - \bar{e}_x \bar{e}_y)^2$$

The corresponding tank trajectories in fig. 2 correspond to the three forcing types schematically depicted in figure: (a) the longitudinal elliptic forcing whose the major axis belongs to O_x , (b) the oblique elliptic forcing when the major axis has $0 < \gamma \leq \pi/4$ to the O_x -axis, and (c) the circular orbital forcing, which is the limiting case of (a) and (b). Here, $A = \sqrt{a^2 + \bar{a}^2}$, $B = \sqrt{b^2 + \bar{b}^2}$ and the solid lines imply forcing in fig. 2 (c).

Fig. 3. The amplitude response curves in the

$(\sigma / \sigma_1, A, B)$ space for the circular orbital tank the stable steady-state sloshing. Specifically, the loop-type curves correspond to neatly-standing waves but the hard-spring type response curve reflects the swirling wave mode.

We conducted a numerical study of the steady-state waves and their stability for a range of the input parameters. The results are illustrated in fig. 3 for the circular type forcing from fig. 2(c). Here, the solid lines correspond to the stable sloshing but the dashed ones imply the hydrodynamic instability. The numerical analysis shows a paradox: even though the tank is exposed to a circular forcing and the only swirling is expected as in [2] for the circular base tank, there is a frequency range where the nearly standing (squares-like, [1]) sloshing occurs. This is because of the loop-type response curves in fig. 3. The nearly-standing waves are expected for the forcing frequencies in the left of σ_1 . The aforementioned paradox requires an experimental conformation and deducted model tests

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